ANSWER

1a. Which of the two event releases more energy? Write E_{SN} or E_{GW}.

$35M_{\odot}C^2 + 30M_{\odot}C^2 = 62M_{\odot}C^2 + E_{GW}$ [1 point]

$3M_{\odot}C^2 = E_{GW}$ [0.5 points]

$M_{\odot} = 1.989\times10^{30}\ kg$

$E_{GW} = 3(1.989\times10^{30}\ kg)(9\times10^{16}\ \frac{m^2}{s^2})$ [0.5 points]

$E_{GW} = 5.3703\times10^{47} J$ [1 point]

$E_{GW} = 2\times10^{44} J$

GW150914 released approximately 2865 times more energy than a supernova explosion. $E_{SN} << E_{GW}$. [1 point]

1b. Quantify your answer with an $\frac{E_{SN}}{E_{GW}}$ ratio.

Based on Einstein's mass-energy relationship, $E=mc^2$. The relationship is
$\frac{E_{SN}}{E_{GW}} = 3.7\times10^{-4}$ [1 point]

T2: Temperature of the Earth (10 points)

2a.

The solar radiation intensity (I_T) received on Earth is:

$$I_T = \frac{P_s}{4\pi {r_{s-t}}^2} = \sigma T_s^4 \cdot \left(\frac{R_s}{r_{s-t}}\right)^2 \ [\mathbf{1\ point}]$$

with

$$P_s = 4\pi\sigma R_s^2 T_s^4$$

Also, Earth would absorb energy by this rate:

$$P_{abs} = I_T \pi R_t^2 = \pi\sigma T_s^4 \cdot \left(\frac{R_s \cdot R_t}{r_{s-t}}\right)^2$$

With $R_t = 6.4 \times 10^6 m$ as the radius of the planet "disk". **[1 point]**

Then, by the thermal equilibrium, the absorbed radiation would be radiated over the planet's surface:

$$P_{abs} = P_{rad}$$

With

$$P_{rad} = 4\pi\sigma R_t^2 T_t^4$$

$$\pi\sigma T_s^4 \cdot \left(\frac{R_s \cdot R_t}{r_{s-t}}\right)^2 = 4\pi\sigma R_t^2 T_t^4 \quad [\mathbf{1\ point}]$$

$$T_t = T_s \cdot \left(\frac{R_s}{2\, r_{s-t}}\right)^{1/2} = \mathbf{278.58\ K = 5.43\ °C} \quad [\mathbf{1}\ \boldsymbol{point}]$$

This would be very cold but still viable to harbor life.

2b.

The Earth's absorbed radiation, considering the albedo, is:

$$P'_{abs} = 0.7 \cdot \pi\sigma T_s^4 \cdot \left(\frac{R_s \cdot R_t}{r_{s-t}}\right)^2 \ [\mathbf{1\ point}]$$

$$T'_T = T_s\,(0.7)^{1/4} \cdot \left(\frac{R_s}{2\, r_{s-t}}\right)^{1/2} = \mathbf{254.81\ K = -18.34\ \ °C}\ [\mathbf{1\ point}]$$

2c.

If Earth reabsorbs 58% of the 70% reemitted energy, then:

$$T_T'' = T_s\,[0.7 + (0.58 \cdot 0.7)]^{1/4} \cdot \left(\frac{R_s}{2\,r_{s-t}}\right)^{1/2} = \mathbf{285.68\,K = 12.53\ {}^{\circ}C} \quad [\mathbf{4p}]$$

T3: Mars (10 points)

3a.

Since the path between A and B is parabolic, the total energy of the spacecraft is zero,

$$E_{AB} = 0, \quad \leftrightarrow \quad \varepsilon = 1. \qquad \text{[1 point]}$$

So, when you get to point B, then

$$E = \frac{1}{2}mv_B^2 - \frac{GMm}{r_B} = 0; \qquad \text{[1 point]}$$

From where

$$v_B = \sqrt{\frac{2GM}{r_B}} \qquad \text{[1 point]}$$

Replacing with available values

$$v_B = \sqrt{\frac{2 \times 6{,}67 \times 10^{-11} \times 6{,}4 \times 10^{23}}{6{,}8 \times 10^{6}}} \approx 3{,}54 \times 10^{3}\ \text{m/s}$$

For part b, there are two possible solutions

3b. Solution A

First method: Immediately after breaking the total energy and the angular momentum change since the braking force is tangential, but along the elliptical path BC the values with which it passes through are conserved and take on during the entire journey. C; then applying conservation of energy between points B' (immediately after braking) and C as well as conservation of the angular momentum between B' and C, we have

$$\frac{1}{2}mv_{B'}^2 - \frac{GMm}{r_B} = \frac{1}{2}mv_C^2 - \frac{GMm}{R_M} \qquad \textbf{Equation. 1} \qquad \text{[1 point]}$$

$$L = mv_{B'}r_B = mv_C R_M \qquad \textbf{Equation. 2} \qquad \text{[1 point]}$$

From these two equations we solve for v_C and it's result is used to calculate the total energy in C which gives us:

In both methods, this and next, two points for getting the equation and two points for calculation + negative sign + unit.

Second method, solution B, , direct and simple: It is known that in an elliptical path the total energy depends on the semi-major axis in the form:

Equation. 3 [2 point]

Since we know the major axis, we know the total energy. Applying in (eq. 3) the given values are obtained

-2,10 J [2 point]

3c.

We also have two methods; either by conservation of energy, or by conservation of angular momentum. More direct by the first way, like this:

$$E_{total\ C} = E_{elipse} = \frac{1}{2}mv_C^2 - \frac{GMm}{R_M}$$ **Equation. 4** [1 point]

Using the expression (eq.3) in (eq. 4) and solving for v_C

$$v_C = \sqrt{\frac{2GMr_B}{R_M(R_M + r_B)}}$$ [1.5 point]

Note that this result is independent of the mass of the ship. Replacing with the given values (or using previous results)

[1.5 point]

T4: ALMA - Calculating photons [10 points]

4a.

To get the number of photons per second we have to multiply the Flux by the area of the dish, to know the incoming energy per second, and divide this by the energy of a photon. For $\lambda_1 = 0.32\ mm = 3.2\times10^{-4} m$:

Energy of a photon: $E = \frac{hc}{\lambda} = 6.2076 \times 10^{-22} Jules$ [1 point]

Area of the disk: $A = \pi R^2 = 113.09\ m^2$

Number of photons per second:

$$n_1 = \frac{flux \times A}{E} \approx 1820\ photon/s$$ [1 point]

4b.

Same calculation, just changing the energy of the individual photons:

For $\lambda_2 = 8.6\ mm = 8.6 \times 10^{-3} m$

Energy of a photon: $E = \frac{hc}{\lambda} = 2.3098 \times 10^{-23} Jules$ [1 point]

Number of photons per second:

$$n_2 = \frac{flux \times A}{E} \approx 4900 photon/s$$ [1 point]

4c.

Spatial resolution of a single telescope is given by:

$$\theta = 1.22 \frac{\lambda}{D}$$

where D represents the diameter of the dish

This value will be given in radians, so it must be converted to arcsec afterwards

For a frequency of 74.9 GHz, the corresponding wavelength is:

$$\lambda = \frac{c}{f} = 4\ mm$$ [1 point]

And the spatial resolution:

$$\theta = 1.22\frac{4mm}{12\ m} \approx 83.93\ arcsec$$ [1 point]

4d.

For an array the correct expression is:

$$\theta = \frac{\lambda}{B}$$ [1 point]

being B the longest baseline in the array.

So in this case:

$$\theta = \frac{4mm}{16km} \approx 0.0516\ arcsec$$ [1 point]

4e.

The SEFD is a characteristic flux of a system, found by dividing the characteristic energy associated to the so-called temperature of the antenna by its effective area:

$$SEFD = \frac{2kT_{sys}}{A_e}$$ [1 point]

As no additional information is given about the effective area, the actual physical area of the array should be used:

$$A = 54\left(\pi\times6^2\right) + 12\left(\pi\times3.5^2\right)\approx6569\ m^2$$ [0.5 point]

Substituting the Boltzmann constant, the given temperatura of the antenna, and converting the answer to Jansky we get:

$$SEFD\approx290.46\ Jy$$ [0.5 point]

NOTES:

Question	We indicate the answer must be:	Tolerance
4.a	Approximated to the nearest integer	+/- 10 photons
4.b	Approximated to the nearest integer	+/- 10 photons
4.c	2 digit of precision	[83.0 , 85.0]
4.d	2 digits of precision	0.05 exact
4.e	2 digit of precision	[289.0 , 291.0]

Solution.

5a. Pressure inside the flux tube = Pressure outside the flux tube (surroundings)

(Magnetic pressure + Gas pressure inside) = Gas pressure outside

Particularly,

$$\frac{B_0^2}{2\mu_0} + P_{gas_{in}} = P_{gas_{out}} \qquad \text{[2 points]}$$

$$\frac{B_0^2}{2\mu_0} = P_{0_{out}} - P_{0_{in}} \qquad \text{Eq. (1)}$$

Similarly, for a particular height z,

$$\frac{B(z)^2}{2\mu_0} = e^{-z/H}\left(P_{0_{out}} - P_{0_{in}}\right) \qquad \text{eqn (2)} \qquad \text{[2 points]}$$

Dividing equation (2) to (1),

$$\frac{B(z)^2}{B_0^2} = e^{-z/H} \qquad \text{[1 point]}$$

To finally get

$$B(z) = B_0 e^{-z/2H} \qquad \text{[1 point]}$$

5b.

At $z = H$, $B = B_0 e^{-z/150}$

$$0.03 = 0.3\ e^{-\frac{z}{150}}$$

[1.5 points]

$$z = 150\times \ln ln\ 10$$

$$z = 150\times 2.301$$

[1.5 points]

$$z = 345.2\ km$$

[1 points]

SOLUTIONS2

6a.

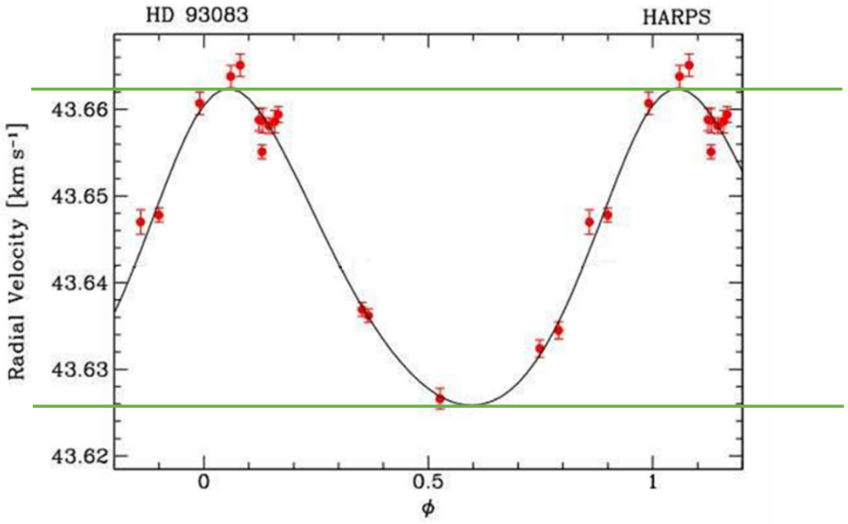
$$\lambda_{max} = \frac{2.898\times10^{-3} m\cdot K}{T_{eff}(K)} = \frac{2.898\times10^{-3} m\cdot K}{4995\ K} = 5.8\times10^{-7} m = 580\ nm$$ [1 point]

6b. With the parallax doable to find the distance from Earth:

$$d = \frac{1}{pllx\ (arcsec)} = \frac{1}{0.035\ arcsec} = 28.57\ pc$$ [1 point]

$$M = m_v - 5\ log(d[pc]) + 5 = 8.3 - 5\ log(28.57) + 5 = 6.02$$ [1 point]

6c. [2 point]



Each division on Y has 0.002 kms^{-1}, so the maximum radial velocity is 43.662 kms^{-1} and the minimus is 43.626 kms^{-1}. Then, the mean radial velocity of Macondo is 43.644kms^{-1}.

6d. The tangential velocity of Macondo, from the plot, is its variation from the mean system velocity (vs = 43.662 kms^{-1} - 43.644 kms^{-1} = 0.018 kms^{-1}). The masses of the star and the planet are known so it is only needed to find orbital velocity of Melquiades. But first it is important to convert the mass of the star to kg.

$$0.7M_\odot \times \frac{1.989\times10^{30}kg}{1M_\odot} = 1.4 \times 10^{30}kg$$ [1 point]

$$v_p = \frac{m_s}{m_p} \times v_s = \frac{1.4\times10^{30}\,kg}{7\times10^{26}\,kg} \times 0.02\,km\cdot s^{-1} = 40\,km\cdot s^{-1}$$ [1 point]

Many students will write this as 36 km/s. We can accept that too.

6e. As the motion of the planet is circular, the orbital period is:

$$T = \frac{2\pi}{v_p} \times a$$ [1 point]

being ***a*** the distance to the central star.

With the 3rd Kepler's law, the orbital period is $T^2 = \frac{4\pi^2}{Gm_s} \times a^3$ [1 point]

Then, combining both expressions of *T* is possible to find *a*

$$\left(\frac{2\pi}{v_p} \times a\right)^2 = \frac{4\pi^2}{Gm_s} \times a^3$$

$$\left(\frac{2\pi}{v_p}\right)^2 \frac{Gm_s}{4\pi^2} = a$$ [1 point]

$$a = 7.21\times10^{10}\,m = 0.48\,au$$

With the previous solution is able to find the orbital period:

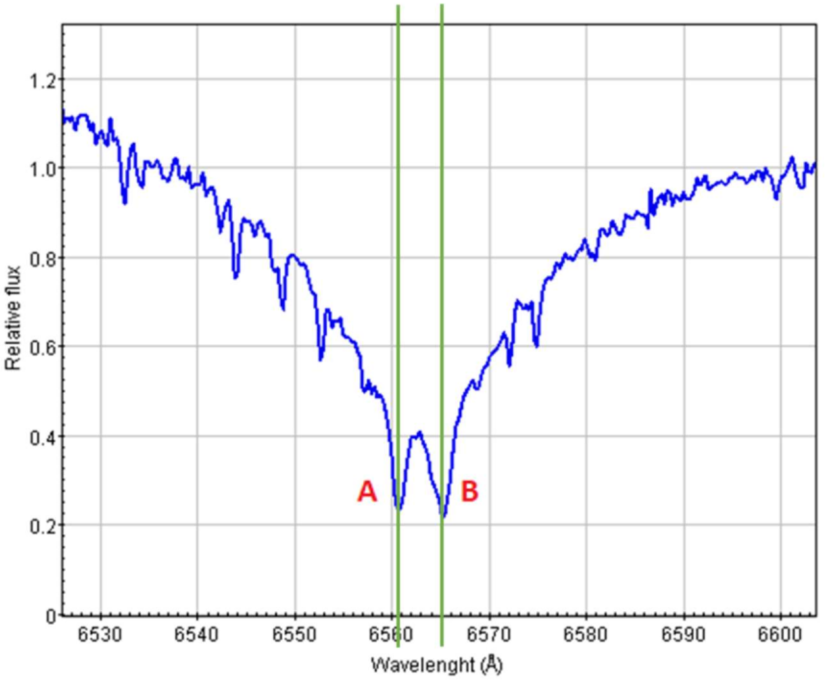
$$T = \frac{2\pi}{v_p} \times a = \frac{2\pi\times(7.21x10^{10}m)}{36000\,m\cdot s^{-1}} = 1.258 \times 10^{7}s = 145.65\,days$$ [1 point]

Finally, with the parallax doable to find the distance from Earth:

$$d = \frac{1}{pllx\ (arcsec)} = \frac{1}{0.035\ arcsec} = 28.57\ pc = 8.82 \times 10^{17}\ m$$ [1 point]

SOLUTION

7a. By the plot is possible to estimate the wavelength of both Hα lines and later find the velocity of each star by Doppler.



$$v_A = \frac{6560.4Å - 6562.8Å}{6562.8Å} \times 3 \cdot 10^5 kms^{-1} = -109.709 kms^{-1}$$ (2 point)

$$v_B = \frac{6565.4Å - 6562.8Å}{6562.8Å} \times 3 \cdot 10^5 kms^{-1} = 118.852 kms^{-1}$$ (2 point)

So, Menkalinan A is approaching and Menkalinan B is moving away from us. (1 point)

7b. Applying the 3rd Kepler's law, the total mass of the system is :

$$m_t = \frac{4\pi^2}{G} \times \frac{\left(0.00339'' \times \frac{1rad}{206265''}\right)^3 \cdot \left(81.1ly \cdot \frac{9.46 \cdot 10^{15}m}{1ly}\right)^3}{\left(3.96d \cdot \frac{86400s}{1d}\right)^2}$$
$$= 1.014 \cdot 10^{31}kg = 5.1M\odot$$

(3 point)

7c.

$$\boldsymbol{m_t = m_1 + m_2 = 1.026 \cdot m_2 + m_2 = 2.026m_2 = 1.014 \cdot 10^{31}\, kg}$$

$$\boldsymbol{\frac{m_1}{m_2} = 1.026}$$

$$\boldsymbol{m_2 = \frac{1.014 \cdot 10^{31}}{2.026} = 5.005 \cdot 10^{30} = 2.52\, M_\odot}$$ (1 point)

(1 point)

$$m_1 = 1.026 \cdot m_2 = 2.59M\odot$$ (1 point)

7d. Using the mass/luminosity relation given:

$$\frac{L_1}{L\odot} \approx \left(\frac{m_1}{M}\right)^{3.5} = \left(\frac{2.59M}{M}\right)^{3.5} = 27.96 = 27.96L_\odot$$ (1 point)

$$\frac{L_2}{L\odot} \approx \left(\frac{m_2}{M}\right)^{3.5} = \left(\frac{2.52M}{M}\right)^{3.5} = 25.40 = 25.4L\odot$$ (1 point)

ANSWER T8: IOAA Logo: (15 points)

8.1.

Assuming R=6378 km (as per the table of constants)

Δx = R*[(4,36,18)-(4,35,30)]*π/180 = 1.4842 km

Δy = R*[(74,3,19)-(74,3,15)]*πi*cos(4,35,54)/180 = 0.1237 km

Δ z = (3.100 - 3.296) = -0.196 km **[2 pt]**

d = $\sqrt{(x^2 + y^2 + z^2)}$ = 1.502 km **[1 pt]**

8.2 Estimate the angular separation (in degrees) between Guadalupe (point 2) and Monserrate (point 3) as observed from the National Astronomical Observatory of Colombia (point 1). Taking point 1 as vertex.

Using same method as part a:

d_{1-2} = 2.633 km **[2 point]**

d_{1-3} = 2.533 km **[2 point]**

Using Cosine rule for spherical triangle,

cos A = (b^2 + c^2 - a^2)/2bc = 0.8345 **[1 point]**

A = 33.4° **[1 point]**

8.3

$$\propto = tan^{-1}\left(\frac{-sin\beta sin\epsilon + cos\beta cos\epsilon sin\lambda}{cos\beta cos\lambda}\right)$$ **[1 pt]**

α= 12° 43’ 3” = $0^h\ 50^m\ 52^s$ **[2 pt]**

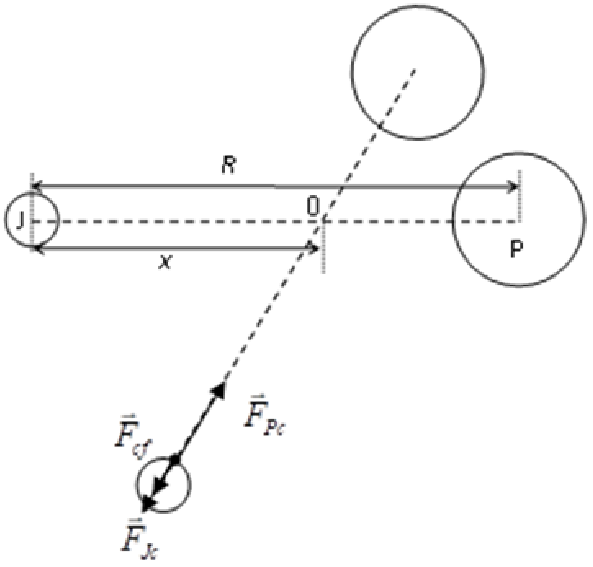
$$\delta = sin^{-1}(sin\beta cos\epsilon + cos\beta sin\epsilon sin\lambda)$$ **[1 pt]**

δ= 1° 33’ 43” **[2 pt]**

ANSWER TQ9: Pluto Satellites (15 points)

[1 point]

If the student makes this graph they will be recognized



Since both bodies move about a common center of mass, point 0, see fig. R-distance between the centers of Pluto and Jaron. x-distance from the center of mass of the system to Jarón (radius of the orbit of Jarón in this system). Jaron's motion through a circular orbit of radius x is given by the equation.

$$G\frac{Mm}{R^2} = m\,\omega^2 x$$ **Equation 1** [2 point]

F_{pc} :force exerted by Pluto on a point on the surface of your satellite.

F_{jc}: force exerted by Jaron on a body on its surface.

F_{cf}: centrifugal force acting on the body that is on the surface of

Charon, (this is a non-inertial frame of reference, it is rotating with respect to 0).

According to the second law for the point on which they are represented the forces remain.

$$g_1 = G\frac{m}{r^2} - G\frac{M}{(R-r)^2} + \omega^2(x-r)$$ **Equation 2** [2 point]

Where m-mass Charon, M-mass Pluto, r radius of Charon. For the other point that is at the other end of Charon it remains:

$$g_2 = G\frac{m}{r^2} - G\frac{M}{(R-r)^2} + \omega^2(x-r)$$ **Equation 3** [2 point]

The difference between both equations is

$$\Delta g = GM[\frac{1}{(R-r)^2} + \frac{1}{(R-r)^2}] - 2\omega^2 x)$$ **Equation 4** [2 point]

Taking into account (1) and representing, $\beta = \frac{r}{R}$, then

$$\Delta g = 2G\frac{M}{R^2}[\frac{1+\beta^2}{(1-\beta^2)^2}] \approx 6G\frac{M}{R^2}\beta^2$$

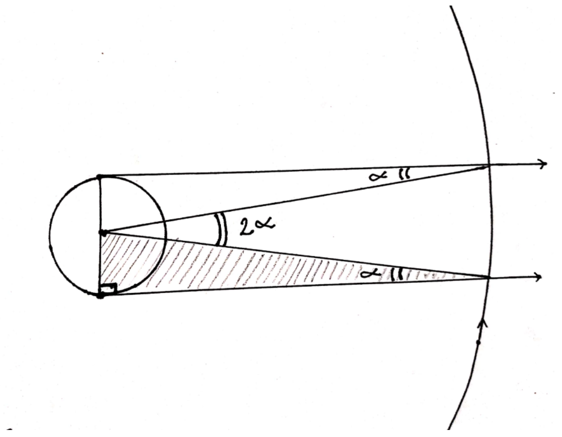
[4 point]

$$\frac{\Delta g}{g} = \frac{6G\frac{M}{R^2}\beta^2}{G\frac{m}{r^2}} = 6\frac{M}{m}\beta^2 = 4 \cdot 10^{-5}$$

[2 point]

SOLUTIONS [15 points]

10a. The maximum duration occurs if Earth passes exactly along the diameter of the Sun. Now consider the following figure, where Sun rays are depicted as traveling to the right towards the hypothetical distant observer (reason why they can be considered parallel):



The arc along the circumference associated to the transit, 2α, can be easily found looking at the shaded triangle:

$$\alpha = sin^{-1}\left(\frac{R_{\odot}}{R_{orb}}\right)$$ [2 points]

being R_{orb} the orbital radius of the Earth. Now the time of the transit can be found by considering the angular velocity of the Earth, for instance by means of the following proportionality relations:

[2 points] $$\frac{t_{tr}}{T_{orb}} = \frac{2\alpha}{2\pi} = \frac{sin^{-1}\left(\frac{R_{\odot}}{R_{orb}}\right)}{\pi}$$

Substituting with the known values for these quantities we get, i.e.,

$$t_{tr} \sim 12.94\, h \;=\; 12h\, 56h$$

[1 points]

10b. First of all it must be said that the minimum orbital period is obtained if we assume that the transit occurs along the diameter of the star. In other cases, the planet would be crossing a shorter path in front of the star during the same time, meaning that it would have a smaller angular velocity and therefore a longer period.

That said, it means that we can resort to the same expression of literal a):

Yet this time we need 2 additional elements:

* Using the small angle approximation:

$$sin(\alpha) \sim \alpha$$ [1 points]

* Invoking the full expression for the orbital period:

$$T_{orb} = \frac{2\pi}{\sqrt{GM_*}} R_{orb}^{3/2}$$ [1 points]

Combining these results, we get:

$$\frac{t_{tr}}{\frac{2\pi}{\sqrt{GM_*}} R_{orb}^{3/2}} = \frac{\frac{R_\odot}{R_{orb}}}{\pi}$$ [2 points]

And solving for R_{orb}:

$$R_{orb} = \frac{GM_* t_{tr}^2}{2^2 R_*^2}$$ [2 points]

Finally putting this last result back into the expression for the orbital period:

$$T = \frac{\pi G M_*}{4R^3} t_{tr}^3$$ [2 points]

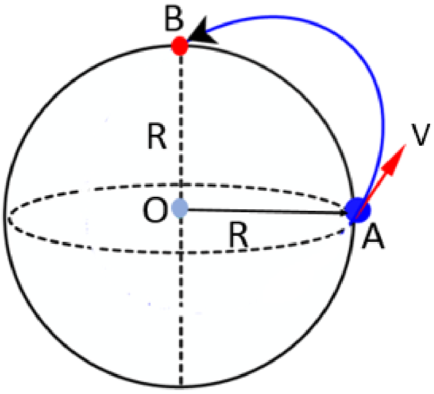
At this point, just a numerical evaluation is needed, though a more elegant solution can be achieved by means of scaling relations by noting that this very same expression must be true in the case of the Earth-Sun system as seen from far away. Having that 31 minutes is the 4% of 12.94h:

$$T = 365.25 \cdot \frac{0.1}{0.1^3} \cdot 0.04^3 \sim 2.34\ días \sim 2d\ 8h\ 10m$$ [2 points]

These values were inspired by the planetary system Trappist 1.

ASWER TQ11: Minimum velocity of a projectile (15 points)

11.1



[1 point]

$$EM = -\frac{GMm}{R} + \frac{1}{2}mv^2$$ **[1 point]**

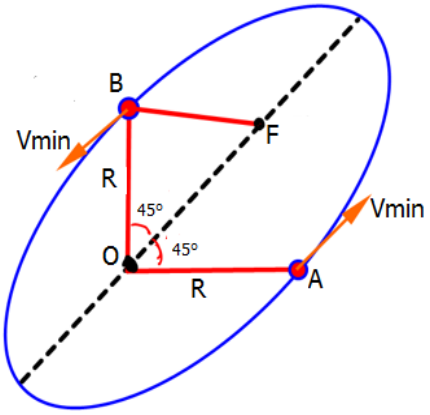
At points A and B, the v is the same; so, if v is minimum, EM is minimum:

We know that the mechanical energy for an elliptical orbit is given by:

$$EM = -\frac{GMm}{2a}$$ **[1 point]**

where 2a is the length of the major axis

Since EM must be minimal, then 2a must be minimal **[1 point]**



[1 point]

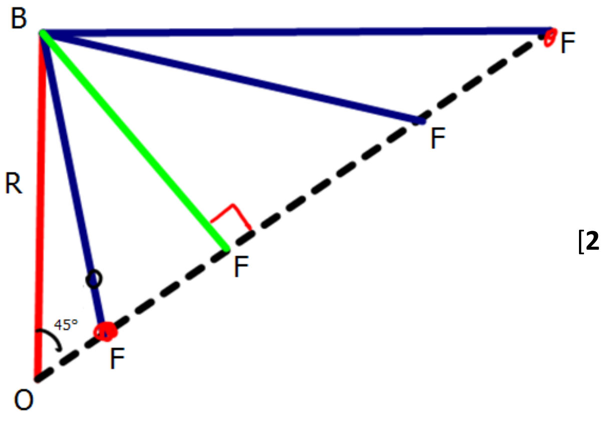
O: Center of the Earth and one of the focus of the ellipse.

F: the other focus of the ellipse

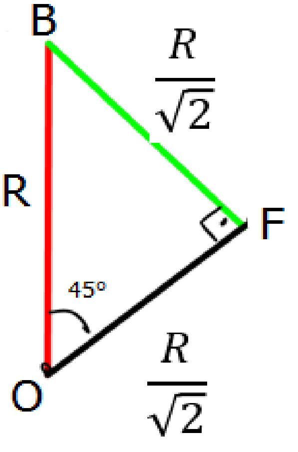
$$OB + BF = 2a$$ **[1 point]**

$OB + BF = 2a$, must be minimal

F, takes an arbitrary position, so to minimize, BF must be perpendicular to OF



[2 points]



[1 point]

$$OB + BF = R + \frac{R}{\sqrt{2}}$$ **[1 point]**

$$2a = R + \frac{R}{\sqrt{2}}$$

This is the minimum value it can take

Correspondingly, the expression for the minimum velocity is

$$v = \sqrt{\frac{2GM}{(1+\sqrt{2})R}}$$ **[1 points]**

Using the M and R values for the Earth, the minimum velocity is

$$v = 7199.07\ \frac{m}{s}$$ **[1 points]**

11.2

Furthermore, $OF = e \cdot a$, where *e* is the eccentricity

$$\frac{R}{\sqrt{2}} = e\,\frac{R}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$ **[2 points]**

$$e = \frac{\sqrt{2}-1}{2} = 0.207$$ **[1 point]**

ANSWER T12:

12.1. Solution

For a satellite that describes a uniform circular motion of radius r around the Sun and mass M, we have:

$$v = [R_{Hodograph}] = \sqrt{\frac{GM}{r}}$$ (1 point)

12.2. Solution

$$\vec{a} = -\frac{Gm}{r^2}\hat{r}$$ (1 point)

$$L = mr^2\omega$$ (1 point)

$$\rightarrow \ \vec{a} = -\frac{GMm\omega}{L}\hat{r} = \frac{\Delta \vec{v}}{\Delta t}$$

$$\frac{GMm}{L}\frac{\Delta\theta}{\Delta t} = \left|\frac{\Delta v}{\Delta t}\right|$$

$$\Delta v = \pm\frac{GMm}{L}\Delta\theta$$ (2 point)

12.3. Solution

$$L = mv_cR$$ (1 point)

$$k_c = \frac{GMm}{L} = v_c = \sqrt{\frac{GM}{R}}$$ (1 point)

12.4. Solution

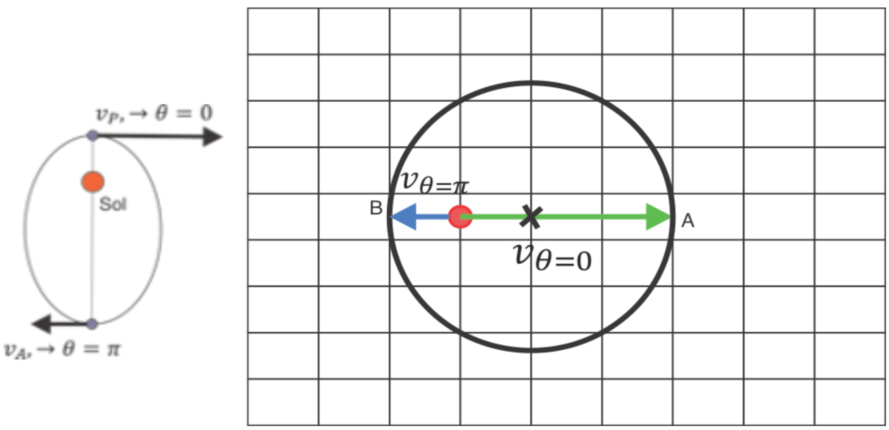
In the following scheme, the homographic circumference has been constructed as follows: The velocity vector is drawn in arbitrary units (3) corresponding to the velocity in the perihelio that is $v_{(\theta = 0)}$, its head determines the point A. Then the velocity vector is drawn diametrically opposite in the aphelion, that is, $v_{(\theta = \pi)}$ also in arbitrary units. Its head determines point B. The hodograph must pass through points A and B. Therefore, the radius of the hodograph must be equal to:

$$Radio_{hod} = \frac{v_{(\theta=0)} + v_{(\theta=\pi)}}{2}$$ (1 point)

And the "distance" *d* from the Sun to the center of the circumference will be:

$$d = Radio_{hod} - v_{(\theta=\pi)} = \frac{v_p - v_a}{2}$$ (1 point)

where *P*, is perigee and *a* is apogee.



(2 point)

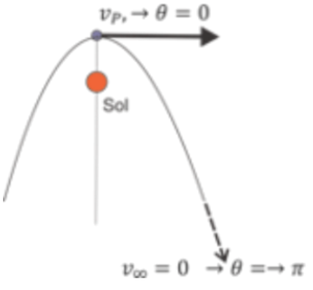
12.5. Solution

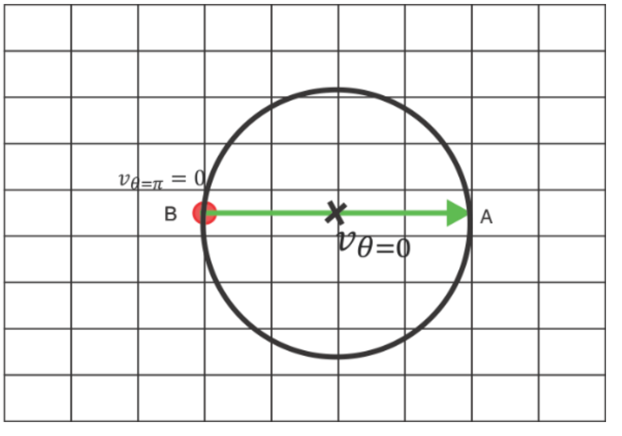
In the following scheme, the homographic circumference has been constructed as follows: The velocity vector is drawn in arbitrary units (4) corresponding to the velocity in the perihelio that is $v_P = v_{\theta=0}$, its head determines the point A. Then the velocity vector is drawn diametrically opposite in the aphelion, that is, $v_{(\theta = \pi)} = 0$. Its head determines the point B that coincides with the Sun. The hodograph must pass through points A and B. Therefore, the radius of the hodograph must be equal to:

$$Radio_{hod} = \frac{v_p}{2}$$ (1 point)

And the distance "*d*" from the Sun to the center of the hodograph circumference

$$d = Radio_{hod} = \frac{v_p}{2}$$ (1 point)





(2 point)

T13: Lucy: The First Mission to the Trojan Asteroids (15 points)

Solutions a.

A single electron deposits energy in the CCD, as follows:

$E_{deposited}$ = stopping power x ρ_{si} x thickness

$$E_{deposited} = 3.012\frac{MeV\,cm^2}{g} \times \frac{2.34g}{cm^3} \times 0.06cm = 422.9\ keV$$ (4 points)

Since an electron with 15 MeV energy deposits 422.9 keV, we must calculate the number of pairs electron/hole

$$422.9keV \times \frac{1}{2.36eV} = 1.79x10^5\frac{e}{h}$$

(4 points)

How many pixels will be able to excite $1.79x10^5$ pairs of e/h?

$$1.79x10^5 x\frac{1}{250} = 716\ pixels$$ (2 points)

Solutions b

The number of electrons entering the detector area is:

(1 points)

$flux \times area \times t$

$$\frac{600\ e}{cm^2 s} \times [1.3 \times 1.3]\,cm^2 \times 0.03s = 30.42e^-$$ (1 points)

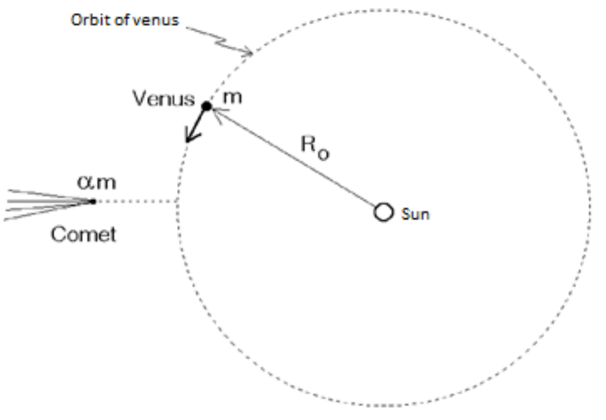
30.42 electrons enter the detector area, and we know that one electron excites 716 pixels, the total number of excited pixels is:

$$716\frac{pixels}{electron} \times 30.42\ electrons = 2.18 \times 10^4 pixels$$ (1 points)

If the CCD has a total of 1024x1024 pixels = $1.048x10^6 pixels$ then the total percentage of pixels that will be driven in a single image is ~ 2.07% (1 points)

TQ14: Formation of the Vennus (35 points)

A comet of mass αm is heading ("Falls") radially towards the Sun. It is known that the total mechanical energy of the comet is zero. The comet crashes into Venus, whose mass is m. We further assume that the orbit of Venus, before the collision, is circular with radius Ro. After the crash, the comet and Venus form a single object, called "Vennus".



14.1. Find the expression for the orbital speed, v_o, of Venus before the collision.

[1 point]

14.2. Find an expression for the total mechanical energy of Venus in its orbit before colliding with the comet. [1 point]

14.3. Find an expression for the radial velocity, v_r, the angular momentum, L, and angular velocity, ω, of Venus immediately after the collision. [10 points]

14.4. Find an expression for the mechanical energy of the combined object "Vennus" and express it in terms of energy before the collision (E_i) and α. [5 points]

14.5. Show that the post-collision orbit of "Vennus" is elliptical and determine the semi-major axis of the orbit. [5 points]

14.6. Determine if the year for the "Vennusians" has been shortened or lengthened because of collision with the comet. [3 points]

14.7. What should be the value of α such that the post collision orbit of Vennus would make it crash in the Sun? We will call this as α_c. [5 points]

14.8. A comet with $\alpha = \alpha_c$ collided with Venus. Calculate the percentage change in the magnitude of Venus' velocity (\delta v) and amount of change in the direction of the velocity vector (\delta\theta). [5 points]

Solutions

14.1. Solution

Let M be the mass of the Sun, then, equating the gravitational force with the force Centripetal

$$-\frac{GMm}{R_0^2}\hat{r} = -\frac{mv_0^2}{R_0}\hat{r}$$

$$v_0^2 = \frac{GM}{R_0}$$

[1 point]

14.2. Solution

The mechanical energy of Venus (before collision) is:

$$E_i = -\frac{GMm}{{R_0}^2} + \frac{1}{2}mv_0^2 = -\frac{GMm}{2\ {R_0}^2}$$

[1 point]

14.3. Solution

Since the comet (when far away) moves radially towards the sun, it has no angular momentum (with respect to the origin in the Sun). Then the angular momentum of Venus is the same as Venus

$$L = R_0 m v_0$$

[1 point]

This allows us to find the component angular of the Venus velocity just after the collision. The angular momentum just after the collision is

$$L = R_0(m + \alpha m)v_\theta$$

[2 points]

Since the angular momentum is conserved it follows that

$v_\theta = \frac{v_0}{1+\alpha}$ [2 points]

The conservation of the linear momentum in the radial direction must be realized that the interaction between Venus and the comet are internal forces and, therefore, to calculate the velocity of the comet we can ignore the effect introduced by the interaction between the comet and Venus. The comet has zero energy, so

$K = -U = +\frac{GM\alpha m}{R_0} = \frac{1}{2}\alpha m v_c^2$ [2 points]

v is the velocity of the comet just before the collision ignoring the effect introduced by Venus). It follows that

$v_C^2 = \frac{2GM}{R_0}$ [1 point]

We now apply the conservation of linear momentum along the radial direction

$\alpha m v_C = (m + \alpha m)v_r$ [1 point]

where v_r is the velocity of Vennus just after the collision. It follows that

$v_r = \frac{\alpha}{1+\alpha}v_C$ [1 point]

14.4. Solution

Vennus mechanical energy (we evaluate it just after the collision) is:

$E_f = U + K = -\frac{GMm(1+\alpha)}{R_0} + \frac{1}{2}m(1+\alpha)\left(v_\theta^2 + v_r^2\right)$ [2 point]

$= -(1+\alpha)\frac{GMm}{R_0} + \frac{1}{2}(1+\alpha)\left[2\left(\frac{\alpha}{1+\alpha}\right)^2 + \frac{1}{(1+\alpha)^2}\right]\frac{GMm}{R_0}$ [2 point]

$= -\frac{GMm}{2R_0}\left(\frac{1+4\alpha}{1+\alpha}\right) = E_i\left(\frac{1+4\alpha}{1+\alpha}\right)$ [1 point]

14.5. Solution

¨Vennus¨ orbit is obviously no longer a circle. Since the energy is negative it must, therefore, be elliptical. It has to

$\frac{E_i}{E_f} = \frac{a_f}{a_i} = \frac{a_f}{R_0}$ [2 point]

Here a_i and a_f are the semi-major axes of the orbits of Venus and Vennus, respectively. It follows that

$a_f = R_0\frac{E_i}{E_f} = R_0\frac{1+\alpha}{1+4\alpha}$ [3 point]

14.6. Solution

Using Third law is Kepler, we have

$$\frac{T'}{T} = \left(\frac{a_f}{r_0}\right)^{3/2} = \left(\frac{1+\alpha}{1+4\alpha}\right)^{3/2}$$

the year is shortened.

14.7. Solution

For the 'Vennus' to collide with the Sun, the perihelion radius of post-collision orbit should be

$$\mathrm{r_p} = \mathrm{R}_{\odot} = 6.955 \times 10^8 \mathrm{m}$$

$$= \frac{6.955 \times 10^8}{0.723 \mathrm{x} 1.496 \times 10^{11}} \mathrm{R_0}$$

[1 point]

$$\mathrm{r_p} = 0.00643 \mathrm{R_0}$$

[0.5 points]

$$r_a = R_0$$

[0.5 points]

Hence

$$a_c = (r_p + r_a)/2 = 0.5032R_0$$

[1 point]

$$\frac{a_c}{R_0} = 0.5032 = \frac{1+\alpha_c}{1+4\alpha_c}$$

$$\alpha_c = \frac{1-0.5032}{4\times 0.5032 - 1}$$

$$\alpha_c = 0.4905$$

[2 points]

14.8. Solution

For post-collision orbit,

$$v_\theta = \frac{1}{1+\alpha_c}v_0$$

$$v_r = \frac{\sqrt{2}\alpha_c}{1+\alpha_c}v_0$$

$$v_f = \sqrt{v_r^2 + v_\theta^2}$$

$$v_f = \frac{\sqrt{2\alpha_c^2+1}}{1+\alpha_c}v_0$$

[1 point]

Thus,

$$\delta v = v_0 - v_f$$

$$= \left(1 - \frac{\sqrt{2\alpha_c^2 + 1}}{1 + \alpha_c}\right) v_0$$

$$= (1 - 0.8165)v_0 = 0.1835v_0$$

$$= 0.1835\sqrt{\frac{6.674 \times 10 - 11 \times 1.998 \times 10^{30}}{0.723 \times 1.496 \times 10^{11}}}$$

$$\delta v = 0.1835v_0 = 6.44km/s$$

[2 points]

$$\delta\theta = \tan^{-1}\left(\frac{v_r}{v_\theta}\right)$$

$$= \tan^{-1}(\sqrt{2}\alpha_c)$$

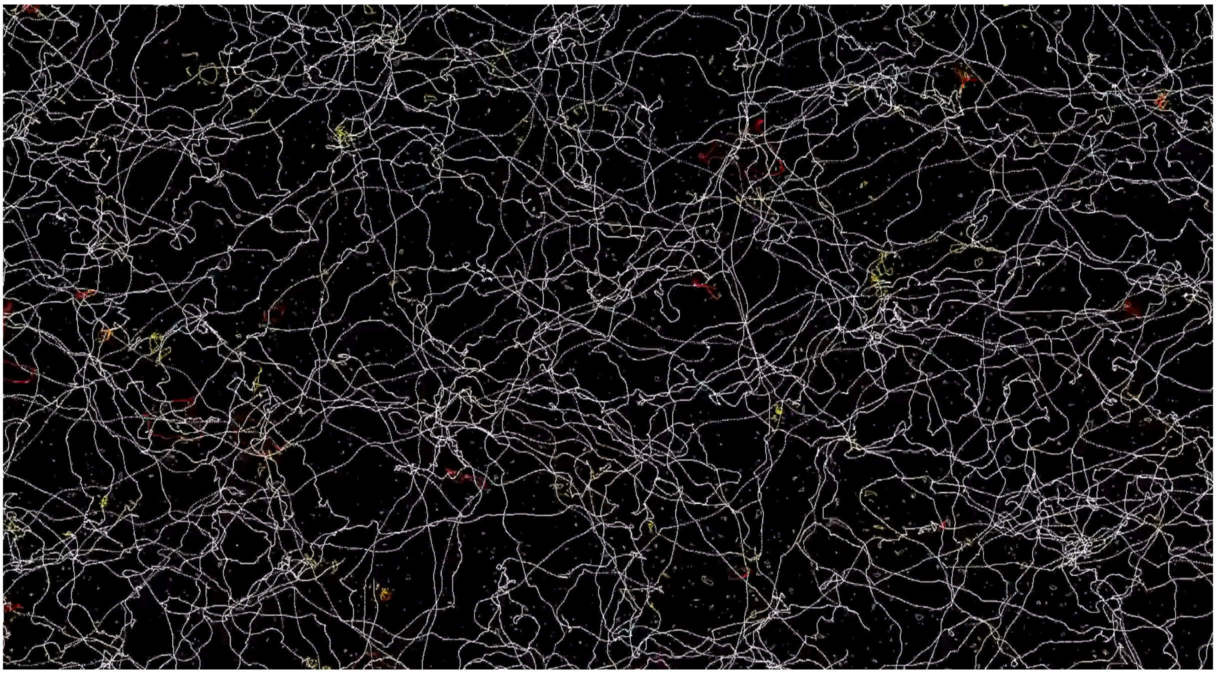
$$\delta\theta = 34.74°$$

[2 points]

TL15: Cosmic Strings (55 points)

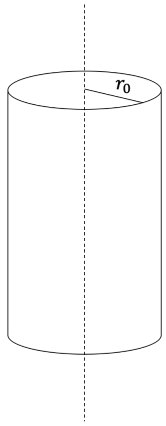
Introduction

According to our current understanding, just after the Big Bang, when the Universe was extremely hot, electromagnetic force, strong nuclear force as well as weak nuclear force were unified as one Grand Unified (GUT) force.
When the Universe cooled down to $T_{GUT} = 10^{29}\ K$, the strong nuclear force decoupled from the electroweak force. Later, when the temperature reduced to $T_{EW} = 10^{15}\ K$, the weak force decoupled from the electromagnetic force. These transitions happened in a rapid succession within a small fraction of a second after the Big Bang. However, it is thought that these phase transitions produced a variety of peculiar objects, called vacuum defects, which may still be observed today.
This question will discuss properties of one such possible type of defect called cosmic strings and their observational effects.



PART A:
Gravitational Field of a Cosmic String (22p)

As a first approximation, let us consider a cosmic string as an infinitely long cylinder of radius r_0 and mass per unit length μ.

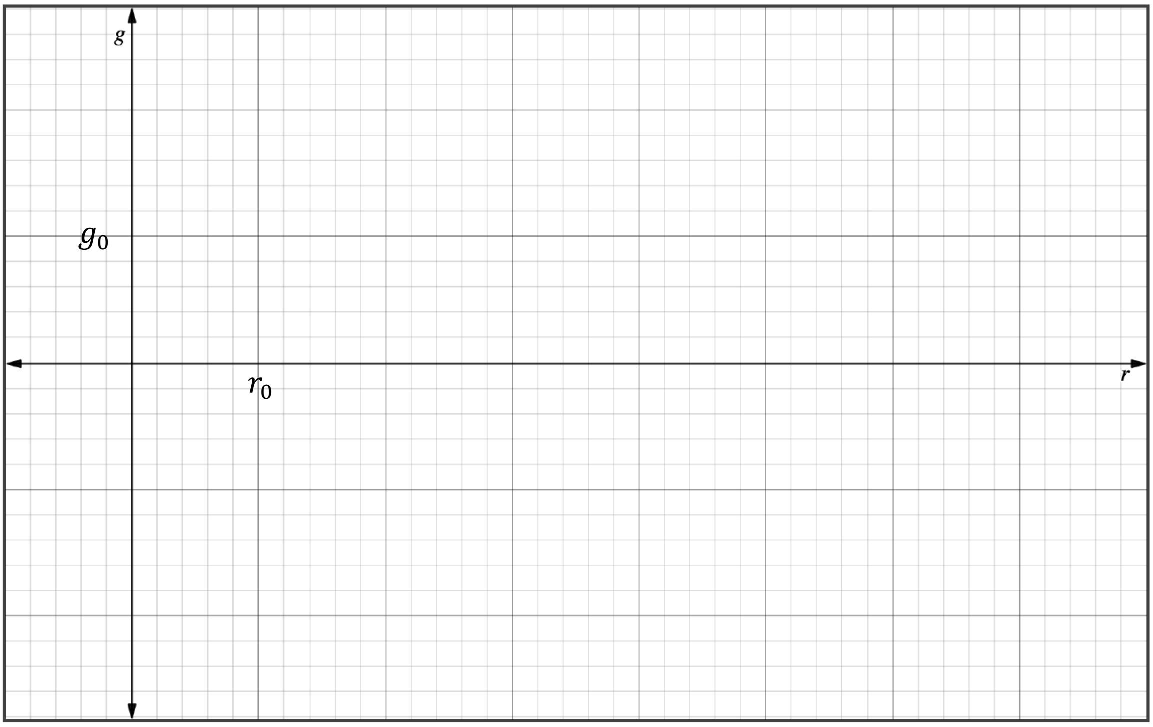


Write an expression in terms of the distance from the center of the string, r, and the constants G, μ, and r_0 for:

A.1 the gravitational field produced by the string, $\vec{g}(r)$. Consider the cases $r_0 < r$, and $r_0 > r$, independently. [6 points]

A.2 $g_0 \equiv |\vec{g}(r_0)|$. [1 point]

A.3 Draw a rough sketch of $\vec{g}(r).\hat{r}$ vs. r, in the figure given in the answer sheet [3 points]



A.4 It is possible to define a stable orbit around a Cosmic String. For circular orbits of radius $R > r_0$ and period τ , the following relation is attained

$$R = A\,\tau^{\alpha},$$

where A and α are constants. Find A and α in terms of G and μ. [4 points]

For the following three questions, consider only the classical regime. A particle has speed v at a distance $R > r_0$from the string.

A.5* Show that the gravitational potential energy of the particle, is [3 $points$]

$$U = Gm\mu\ln(R) + u$$

where u is a constant.

A.6 What is the maximum distance, R_{max}, from the string that the particle can reach? [4 points]

A.7 Is it possible for the particle to escape the gravitational field? Write YES/NO in the answer sheet. [1 point]

***HINT:** You may need the following result

$$\int \frac{dx}{x} = \ln(x) + \gamma,$$

where γ is an unknown constant.

PART B:

Cosmic string as a photon gas (17P)

We now consider a more detailed model: A cosmic string as a very long cylinder of radius r_0 and conducting walls filled with a photon gas in thermal equilibrium at temperature T.

B.1 What is the energy density ρ of the string in terms of T, $\hbar$, k_B and c? **[2 points]**

B.2 The radius r_0 is related with the temperature T as

$$r_0 = \frac{\hbar^{n_1} c^{n_2}}{k_B T},$$

where $\hbar$ is the reduced Planck constant, and c is the speed of light in vacuum, k_B is the Boltzmann constant, and n_1, n_2 are integer numbers. Determine n_1 and n_2. **[4 points]**

B.3 What is the mass per unit length, μ, of the string in terms of ρ and r_0? [**2 points]**

B.4 Express the inequality for the weak field condition

$$\frac{2G\mu}{c^2} \ll 1,$$

only in terms of T and T_{Pl}. **[5 points]**

B.5 Calculate $\frac{2G\mu}{c^2}$ for[3p]

i. $T = T_{EW}$
ii. $T = T_{GUT}$

B.6 Does the weak field condition hold…

i. for T_{EW}? [0.5 points]
ii. for T_{GUT}? [0.5 points]

<u>Note</u>: You may use the following constants

- Stefan-Boltzmann Constant

$$\sigma = \frac{2\pi^5 k_B^4}{15\, h^3 c^2} = \frac{\pi^2 K_B^4}{60\, \hbar^3 c^2}.$$

- the reduced Planck constant

$$\hbar = \frac{h}{2\pi}.$$

- Universal Radiation Constant

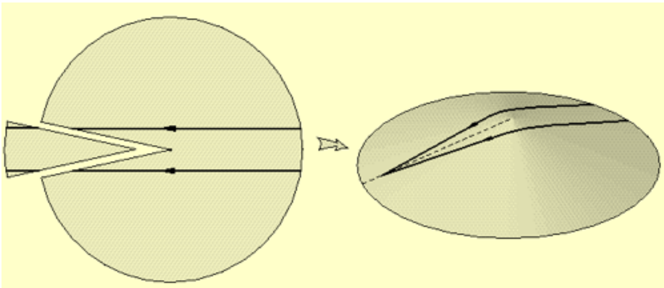
$$a = \frac{4\sigma}{c} = 7.5657 \times 10^{-16} \mathrm{J.m^{-3}.K^{-4}}$$

- Planck Temperature

$$T_{Pl} = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.416784 \times 10^{32} \mathrm{K}$$

PART C:
Gravitational Lensing from cosmic Strings (16P)

So far, in part A and B, we have neglected the internal pressure of the photon gas inside the string. If we include it in our analysis, we need to consider the General Theory of relativity. After solving the Einstein field equations, one finds that the spacetime around a cosmic string is conical as if a narrow wedge were removed from a flat sheet and the edges connected, as shown below.



Source: http://www.ctc.cam.ac.uk/outreach/origins/cosmic_structures_five.php

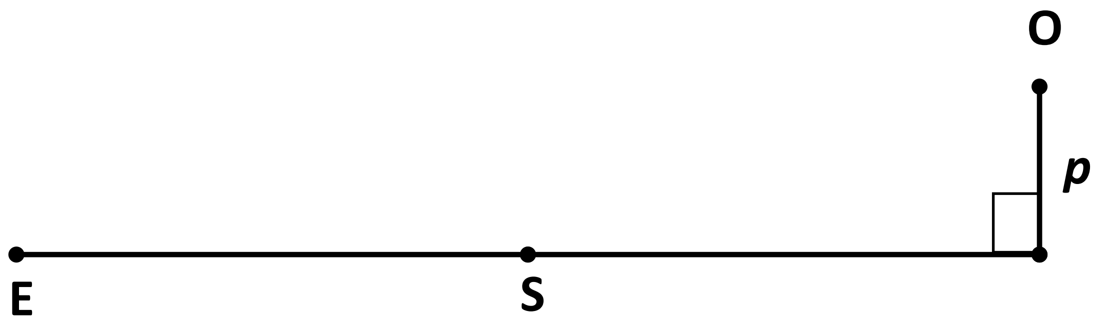
A remarkable result of this model is light deflection by a cosmic string, which leads to the possibility of detection through gravitational lensing. For instance, Strings moving across the line of sight will cause line-like discontinuities in the CMB radiation.

The angle of deflection (in radians) of a light ray coming from a distant quasar (O in the figure below), as the light passes close to a cosmic string (S in the figure below) and eventually reaching an observer on the Earth, (E in the figure below), is

$$\delta\phi = \frac{4\pi G\mu}{c^2}$$

and is independent of the impact parameter, p.

In the figure E and O are in a plane perpendicular to the string. The distance between the observer and the string is D_{ES} and the distance between the observer and the source is D_{OE}.



C.1 Find a condition on the value of the impact parameter p in terms of D_{ES}, D_{OE} and temperature T, for an Earth-based observer to see more than one image of the object O. [6 points]
C.2 In case the observer sees more than one image, what is the angular separation between each pair? Find an expression in terms of D_{ES}, D_{OE} and $\delta\phi$ [6 points]
C.3 If $D_{OE} = 2D_{ES}$, determine the minimum size of an optical telescope needed to resolve this lensing event produced by GUT string. [4 point]

Solution

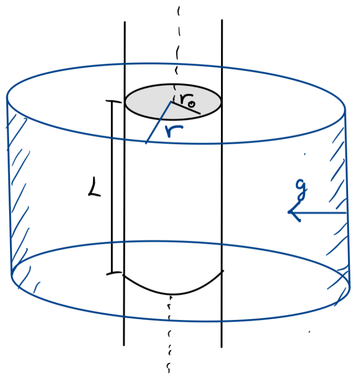
A.1 The gravitational field, which direction is radial

$$\vec{g} = \vec{g}(r) = g(r)\,\hat{r} = g\,\hat{r},$$

can be computed from the Gauss law for the gravitational field:

$$\Phi_g = -4\pi GM,$$

where Φ_g is the flux of the gravitational field through a cylinder of radius r and length L, as shown in the figure, and M is the enclosed mass.



$$\Phi_g = -4\pi GM$$

[2points]

$$2\pi rLg = -4\pi GM$$

$$g = -\frac{2GM}{rL}$$

[1 point]

There are two regimes:

- $r > r_0$:

$$\rightarrow M = \mu L$$

  and

$$g = -\frac{2G\mu}{r}$$

  [1.5 points]

- For the other regime, $r < r_0$:

$$\rightarrow M = \frac{r^2}{r_0^2}\mu L$$

  and

$$g = -\frac{2G\mu}{r_0}\left(\frac{\mathrm{r}}{\mathrm{r_0}}\right)$$

  [1.5 points]

The final answer is

$$\vec{g}(r) = \begin{cases} -\frac{2G\mu}{r_0}\left(\frac{\mathrm{r}}{\mathrm{r_0}}\right)\hat{r}, & r < r_0 \\ -\frac{2G\mu}{r}\hat{r}, & r > r_0 \end{cases}$$

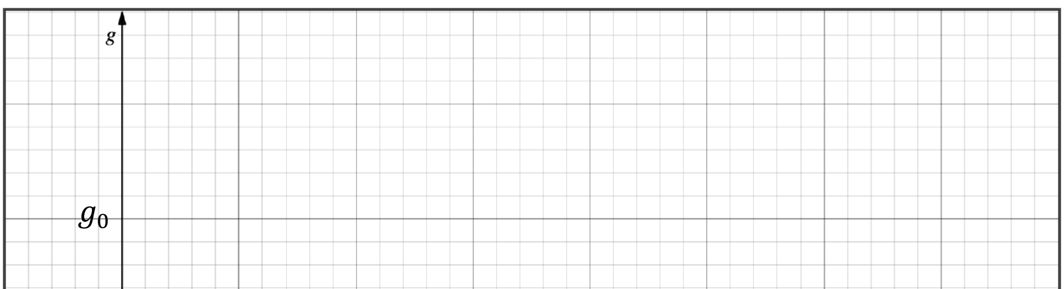
A.2 Writing $\vec{g}(r_0)$, in terms of in terms of G, μ and r_0[1 point]

$$\vec{g}(r_0) = -\frac{2G\mu}{r_0}\hat{r}$$

$$g_0 \equiv |\vec{g}(r_0)| = \frac{2G\mu}{r_0}$$

A.3 Plot $g \equiv \vec{g}(r).\hat{r}$ vs r -> 0.5 point each

- (0,0)
- (1,-1)
- (5,-0.2)
- Linear between (0,0) and (1,-1)
- $\sim \frac{1}{r}$, for $r > r_0$
- Below x axis



A.4 The centripetal acceleration equals the gravitational acceleration, thus

$$\frac{v^2}{R} = \frac{2G\mu}{R} \ \ [1p]$$

$$\frac{2\pi R}{\tau} = \sqrt{2G\mu} \ \ [1p]$$

$$R = \frac{\sqrt{2G\mu}}{2\pi}\tau$$

$A = \frac{\sqrt{2G\mu}}{2\pi}$, [1p] and $\alpha = 1$ [1p].

A.5 The potential energy of the particle is given by

$$U = m\int g(r)dr \ = G m\mu \int \frac{1}{r}dr \ \ [\mathbf{2p}]$$

$$U = Gm\mu \ \ln(r) + u \ [\mathbf{1p}]$$

Where u is a constant that sets the 0 of U.
Usually, U=0 is set at $\mathrm{r} = \infty$, *however for a cosmic string this is not possible.*

We can choose U=0, for example, at the surface of the string $r = r_0$ *(this choice is not relevant!).* Thus

$$u = -Gm\mu \ \ln(r_0)$$

and

$$U = Gm\mu \ \ln\left(\frac{r}{r_0}\right)$$

A.6 The total energy of the particle is conserved, thus:

$$\frac{1}{2}mv^2 + Gm\mu \ \ \ln\left(\frac{R}{r_0}\right) = Gm\mu \ \ \ln\left(\frac{R_{max}}{r_0}\right) [2p]$$

Solving for R_{max}:

$$R_{max} = Re^{\frac{v^2}{2G\mu}} \text{ [2p]}$$

As mentioned before, the answer does not depend on r_0

<u>Note</u>:

$$U = Gm\mu \ln\left(\frac{r}{r_0}\right)$$

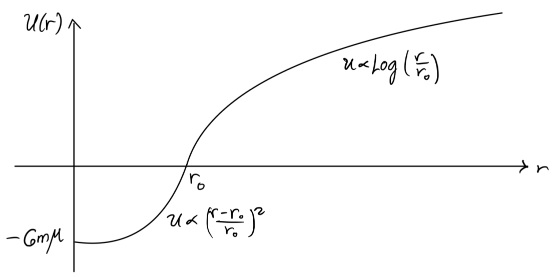
holds only for $r > r_0$.

The potential inside the string is

$$U = Gm\mu\frac{r^2}{r^0} - u',$$

where $u' = -Gm\mu$, *such that* $U = 0$ *at the surface of the string. However, for the solution of the problem it is not necessary to show this.*

$$U = \begin{cases} 2G\mu\left(\left(\frac{\mathrm{r}}{\mathrm{r}_0}\right)^2 - 1\right), & r < r_0 \\ 2G\mu \ \ln\left(\frac{r}{r_0}\right), & r > r_0 \end{cases}$$



A.7 From the previous result,

$$R_{max} = Re^{\frac{v^2}{2G\mu}},$$

we see that it is **not possible** to escape the gravitational field since for any speed, there is always a maximum distance R_{max} **[1p]**

B.1 The energy density is given by

$$\rho = aT^4[\mathbf{1p}]$$

Equivalently:

$$\rho = \frac{4\sigma}{c}T^4 = \frac{\pi^2 k_B^4}{15\,\hbar^3 c^3}T^4[\mathbf{1p}]$$

<u>Note:</u> These result comes from the integration of the spectral energy density given by the Planck law over all frequencies' ν. However it is not required to show this integral.

B.2

$$r_0 = \frac{\hbar^{n_1} c^{n_2}}{k_B T}$$

- r_0 has dimensions of lenght : $[l]$

- $\hbar$ has dimensions of energy $\times$ time: $[\mathrm{e}\ t]$
- c has dimensions of speed: $[l/t]$
- $k_B T$ has dimensions of energy: $[e]$

Then:

$$[l] = \frac{[\mathrm{e}\ t]^{n_1}\left[\frac{l}{t}\right]^{n_2}}{[e]}$$

Since the LHS and the RHS should have the same dimensions we get:

$$[l]: 1 = n_2$$
$$[e]: 0 = n_1 - 1$$
$$[t]: n_1 = n_2[2p]$$

The unique solution is $\mathrm{n}_1 = \mathrm{n}_2 = 1$**[1+1]**

B.3 The energy of a piece of the string of length L is

$$E = \rho\pi r_0^2 L = Mc^2, [\mathbf{1p}]$$

where $M = \mu L$ is the mass of the piece. Solving for μ:

$$\mu = \frac{\rho\pi r_0^2}{c^2}[\mathbf{1p}]$$

B.4 The weak field condition is

$$\frac{2G\mu}{c^2} \ll 1$$

$$2G\frac{\rho\pi r_0^2}{c^2} \ll 1$$

$$2G\frac{\pi}{c^2}(aT^4)\left(\frac{\hbar c}{k_B T}\right)^2 = \frac{2Ga\hbar^2\pi}{k_B^2}T^2 \ll 1[\mathbf{3p}]$$

Where we used: $\mu = \frac{\rho\pi r_0^2}{c^2}$**,** $\rho = aT^4$**,** $r_0 = \frac{\hbar c}{k_B T}$

$$\frac{2Ga\hbar^2\pi}{c^2 k_B^2}T^2 \ll 1$$

$$\frac{2G\hbar^2\pi}{c^2 k_B^2}\left(\frac{\pi^2 k_B^4}{15\ \hbar^3 c^3}\right)T^2 \ll 1$$

$$\frac{2\pi^3}{15}\left(\frac{Gk_B^2}{\hbar c^5}\right)T^2 \ll 1$$

$$\frac{2\pi^3}{15}\frac{T^2}{T_{Pl}^2} \ll 1[\mathbf{2p}]$$

where $T_{Pl} = 1.416784 \times 10^{32}$K *(known as the Planck Temperature)*

Note the numerical factor $\frac{2\pi^3}{15}$ ~ 4.13*, (a 5% error tolerance is accepted)*

B.5 The weak field condition is

$$4.13\left(\frac{T}{T_{Pl}}\right)^2 \ll 1$$

equivalently

$$\sim 2\frac{T}{T_{Pl}} \ll 1[\mathbf{1p}]$$

i. $\frac{2T_{EW}}{T_{Pl}} \sim 2\times\frac{10^{15}}{10^{32}} \approx 1.4\times10^{-17} \ll 1$ [1p]

ii. $\frac{2T_{GUT}}{T_{Pl}} \sim 2\times\frac{10^{29}}{10^{32}} \approx 1.4\times10^{-3} \ll 1$ [1p]

Note: Using the following expresion

$$4.13\left(\frac{T}{T_{Pl}}\right)^2 \ll 1$$

One gets

i. $4.13\left(\frac{T_{EW}}{T_{Pl}}\right)^2 \sim 2.1\times10^{-34} \ll 1$

ii. $4.13\left(\frac{T_{GUT}}{T_{Pl}}\right)^2 \sim 2.1\times10^{-6} \ll 1$

And are also acceptable answers *(a 5% error tolerance is accepted)*

B.6

i. Does it hold for T_{EW}? Yes[0.5]

ii. Does it hold for T_{GUT}? Yes[0.5]

C.1 From the figure, the asymptotic condition for the observer to see a second image is that the light ray travelling from O directly to S should bend and travel along SE. This is the maximum angle of bending.

As all angles are small, we can safely use

$$\sin x \approx \tan x \approx x$$

Now,

$$\frac{p}{SP} < \delta\phi[\mathbf{2p}]$$

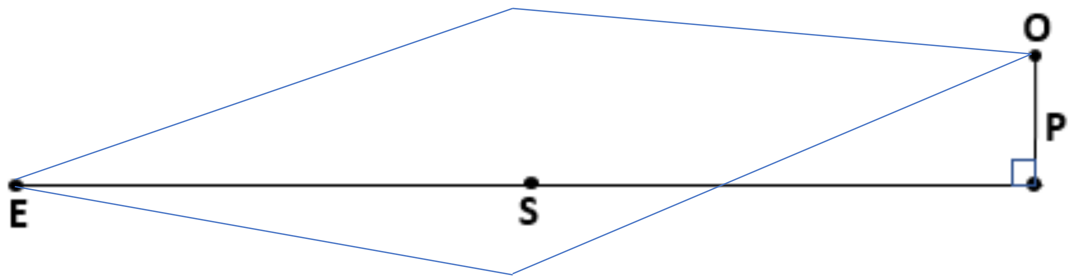
$$< \frac{4\pi G\mu}{c^2}$$

$$< 2\pi\left(\frac{2G\mu}{c^2}\right)$$

$$< 2\pi\left(\frac{2\pi^3}{15}\right)\left(\frac{T^2}{T_{Pl}^2}\right)\ [\mathbf{2p}]$$

And $SP \approx (D_{OE} - D_{ES})$

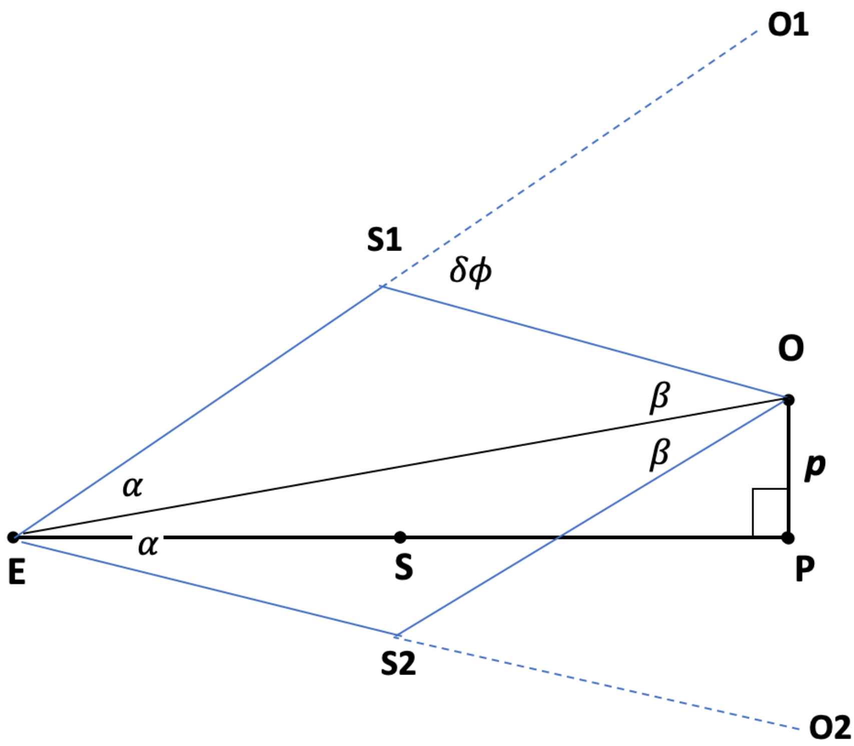
$$p < \frac{4\pi^4}{15T_{Pl}^2}T^2(D_{OE} - D_{ES})\ [\mathbf{2p}]$$



C.2 In the figure, the blue lines represent the bending of two light rays corresponding to the images O1 and O2

Note that $D_{ES} \approx D_{ES1} \approx D_{ES2}$ and $D_{OS} \approx D_{OS1} \approx D_{OS2}$. Thus the angles $S1EO \approx S2EO \equiv \alpha$ and $S1OE \approx S2OE \equiv \beta$. 2α is the angular separation we are looking for

[2 points]



Further, notice

$$\alpha + \beta = \delta\phi \; [\mathbf{1p}]$$

Applying sine law

$$\frac{\alpha}{D_{S1O}} = \frac{\beta}{D_{ES1}} [\mathbf{1p}]$$

we get

$$2\alpha = 2\delta\phi \left(\frac{D_{OE} - D_{ES}}{D_{OE}}\right) \; [\mathbf{2p}]$$

C.3 If $D_{OE} = 2D_{ES}$,

$$2\alpha = \delta\phi = 2\pi\left(\frac{2\pi^3}{15}\right)\left(\frac{T_{GUT}^2}{T_{Pl}^2}\right) \approx 1.29 \times 10^{-5} [\mathbf{2p}]$$

and

$$\delta\phi = 1.22\frac{\lambda}{D}$$

Thus, using $\lambda \in [3 \times 10^{-7}m, 8 \times 10^{-7}m]$

$$D = 1.22\frac{\lambda}{\delta\phi} \in [3.75 \times 10^{-2}m, 7.51 \times 10^{-2}m][\mathbf{2p}]$$

$$D = 1.22\frac{\lambda}{\delta\phi} \in [3.75cm, 7.51cm][\mathbf{2p}]$$

Any answer within this interval is valid.